Q.No→ State and Prove Bessel's inequality in a Hilbert Space.

Ans. - Statement:- If $\{e_i\}$ is any orthonormal set in a Hilbert space H, then

$$\sum |(x, e_i)|^2 \leq \|x\|^2 \quad (1) \text{ for every } x \in H.$$

Proof:- Let $S = \{e_i : (x, e_i) \neq 0\}$. Then, S is either empty or non-empty finite or denumerable.
If S is empty then $(x, e_i) = 0$ for every $e_i$ and we define $\sum |(x, e_i)|^2$ to be zero and obviously $0 \leq \|x\|^2$ for every $x \in H$.

If S is non-empty and finite then S can be written in the form $S = \{e_1, e_2, \ldots, e_n\}$ for some +ve numbers, it is independent of the order in which elements of S are arranged.

The inequality (1) then reduces to

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$$

If S is denumerable then let the elements of S be arranged in a definite order: say $S = \{e_1, e_2, \ldots, e_n, \ldots\}$.
But the results on absolutely convergent series, if $\sum_{i=1}^{\infty} |(x, e_n)|^2$ converges then it converges absolutely and hence every series obtained from this by rearranging its ~~terms~~ terms also converges

and all such series have the same sum. Also $(x, e_i) = 0$ if $e_i \notin S$. We then define,

$$\sum |(x, e_i)|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$$

and hence, $\sum |(x, e_i)|^2$ is a non-negative extended real number, which depends only on the set $S$ and not on the arrangement of members of $S$. In this case (1) reduces to $\sum_{n=1}^{\infty} |(x, e_n)|^2 \leq \|x\|^2$ which is true since all partial sums $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$ (by Bessel's Inequality for finite orthonormal sets).

"This completes the proof"

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Q.No → If $M$ is a proper closed linear subspace of a Hilbert space $H$, then prove that there exists a non-zero vector $z_0$ in $H$ such that $z_0 \perp M$.

Soln. Let $x \in H$ be such that $x \notin M$ and let $d$ be the distance from $x$ to $M$. Then there exists a unique vector $y_0$ in $M$ such that $\|x - y_0\| = d$. We define $z_0$ by $z_0 = x - y_0$ and observe that since $d > 0$, $z_0$ is a non-zero vector in $H$. If $y$ is an arbitrary vector in $M$ we show that $z_0 \perp y$. If $y = 0$ then obviously $z_0 \perp y$. Next, let $y \neq 0$. Then for every scalar $\lambda$, the vector $v = y_0 + \lambda y \in M$, hence we have

$$\|z_0\| = \|x - y_0\| = d = \inf . \{\|x - v\| : v \in M\}.$$
$$\leq \|x - (y_0 + \lambda y)\|$$

$= \|z_0 - \lambda y\|$

and therefore, $(z_0, z_0) \leq (z_0 - \lambda y, z_0 - \lambda y)$

$= (z_0, z_0) - \lambda(y, z_0) - \bar{\lambda}(z_0, y) + \lambda\bar{\lambda}(y, y)$.

i.e. $0 \leq 2\,\mathrm{Re}\,\lambda(y, z_0) + |\lambda|^2 \|y\|^2$ for all scalars

This is possible only if $(y, z_0) = 0$ otherwise, setting

$\lambda = \overline{(y, z_0)}/\|y\|^2$ would give

$$0 \leq -\frac{2|(y, z_0)|^2}{\|y\|^2} + \frac{|(y, z_0)|^2}{\|y\|^2}$$

$$\text{i.e. } 0 \leq -\frac{|(y, z_0)|^2}{\|y\|^2}$$

or, $0 \leq -|(y, z_0)|^2$ which is impossible.

Thus $y \perp z_0$ or $z_0 \perp y$

Thus whether $y = 0$ or $y \neq 0$, we have $z_0 \perp y$. Thus $z_0 \perp M$.

A Characterization theorem for Complete orthonormal set in a Hilbert Space:-

QN→ Let $H$ be a Hilbert Space and let $\{e_i\}$ be an orthonormal set in $H$. Then the following conditions are equivalent to one another

(I) $\{e_i\}$ is Complete

(II) $x \perp \{e_i\} \Rightarrow x = 0$

(III) If $x$ is an arbitrary vector in $H$, Then $x = \sum (x, e_i) e_i$ [Fourier expansion of $x$]

(IV) If $x$ is an arbitrary vector in $H$ then $\|x\|^2 = \sum |(x, e_i)|^2$; [Parseval's equation].

Proof: (I) $\Rightarrow$ (II). Suppose (I) is true. If (II) is false there exists a non-zero vector $x$ in $H$ such that

$x \perp \{e_i\}$. We define $e = \frac{x}{\|x\|}$. Then $\|e\| = 1$.

and, $e \perp \{e_i\}$. Thus $\{e_i, e\}$ is an orthonormal set which properly contains the orthonormal set $\{e_i\}$. This contradicts the completeness of the family $\{e_i\}$. Hence (I) $\Rightarrow$ (II).

(II) $\Rightarrow$ (III). By theorem, $x - \sum (x, e_i) e_i \perp \{e_j\}$.

Hence, (II) implies that

$x - \sum (x, e_i) e_i = 0$, i.e. $x = \sum (x, e_i) e_i$.

Thus, (II) $\Rightarrow$ (III)

(III) $\Rightarrow$ (IV). Assume (III). Then $x = \sum (x, e_i) e_i$

$\therefore \|x\|^2 = (x, x) = \left(\sum (x, e_i) e_i, \sum (x, e_j) e_j\right)$.

$= \sum (x, e_i) \overline{(x, e_i)}$ [$\because$ inner product is continuous]

$= \sum |(x, e_i)|^2$

Thus (III) $\Rightarrow$ (IV)

(IV) $\Rightarrow$ (I). Suppose (IV) is true. If (I) is false i.e. if $\{e_i\}$ is not complete, it is a proper subset of an orthonormal set $\{e_i, e\}$.

Since $e \perp$ each $e_i$ (IV) gives $\|e\|^2 = \sum |(e, e_i)|^2 = 0$, which contradicts that $e$ is a unit vector.

Thus (IV) $\Rightarrow$ (I).

QNo> Show that an orthonormal set in a Hilbert space is linearly independent.

Soln: Let $S = \{e_i\}$ be an orthonormal set in a Hilbert space $H$ and let $\{e_1, e_2, \text{-----}, e_n\}$ be a finite subset of $S$.

Suppose $\alpha_1, \text{-----}, \alpha_n$ are scalars such that

$$\alpha_1 e_1 + \alpha_2 e_2 + \text{----} + \alpha_n e_n = 0 \quad \text{—— (1)}$$

Then for each $K$ where $1 \leq K \leq n$,

$$\left( \sum_{j=1}^{n} \alpha_j e_j , e_K \right) = \sum_{j=1}^{n} \alpha_j (e_j , e_K) = \alpha_K \quad \text{—— (2)}$$

$$\left[ \because (e_j, e_K) = 1 \text{ if } j = K \atop \quad = 0 \text{ if } j \neq K \right]$$

But from (1), $\sum_{j=1}^{n} \alpha_j e_j = 0$. Therefore, from (2), we have,

$$(0, e_K) = \alpha_K$$

$\therefore \alpha_K = 0$ for $K = 1, 2, \text{----} n$.

$\therefore \{e_1, e_2, \text{-----} e_n\}$ is linearly independent.

Therefore, $S$ is linearly independent.

QNo> There exists a norm preserving mapping of $H$ onto $H^*$.

Proof:- Let $y$ be a fixed vector in $H$ and let $f_y$ be a function defined on $H$ by $f_y(x) = (x, y)$ for all $x \in H$. For any $x_1, x_2 \in H$ and for any scalars $\alpha, \beta$ we have.

$f_y(\alpha x_1 + \beta x_2) = (\alpha x_1 + \beta x_2, y)$

$= \alpha(x_1, y) + \beta(x_2, y)$

$= \alpha f_y(x_1) + \beta f_y(x_2)$.

Therefore, $f_y$ is a linear function on $H$.

Moreover, $|f_y(x)| = |(x, y)|$

$\leq \|x\| \, \|y\|$ [By Schwarz inequality]

Therefore, $f_y$ is continuous and thus $f_y$ is a continuous linear functional on $H$ i.e. $f_y$ is an element of the conjugate space $H^*$. Thus each fixed element $y$ of $H$ determines an element $f_y$ of $H^*$.

Now, $\frac{|f_y(x)|}{\|x\|} \leq \|y\|$ for every non-zero $x$ in $H$.

$\therefore \|f_y\| = \sup_{x \neq 0} \frac{|f_y(x)|}{\|x\|} \leq \|y\|$ — (1)

In fact, $\|f_y\| = \|y\|$. This is clear if $y = 0$ for if $y = 0$, then $f_y = 0$, if $y \neq 0$,

$\|f_y\| = \sup\{|f_y(x)| : \|x\| = 1\}$.

$\geq \left| f_y\left(\frac{y}{\|y\|}\right) \right|$

$= \left| \left(\frac{y}{\|y\|}, y\right) \right| = \frac{1}{\|y\|}(y, y) = \frac{\|y\|^2}{\|y\|}$

$= \|y\|$.

Thus, $\|f_y\| \geq \|y\|$ — (2)

From (1) & (2), we have

$\|fy\| = \|y\|$. Thus the correspondence $y \mapsto f_y$ is a norm preserving mapping of $H$ onto $H^*$.